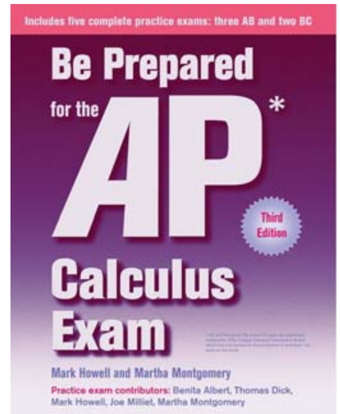

Be Prepared for the AP Calculus Exam

Third Edition

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Mark Howell
Gonzaga High School, Washington, D.C.

Martha Montgomery
Fremont City Schools, Fremont, Ohio

Practice exam contributors:

Benita Albert
Oak Ridge High School, Oak Ridge, Tennessee

Thomas Dick
Oregon State University

Joe Milliet
St. Mark's School of Texas, Dallas, Texas

Skylight Publishing
Andover, Massachusetts

Chapter 10. Annotated Solutions to Past Free-Response Questions

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

2021 AB

AP Calculus Free-Response
Solutions and Notes

Question AB-1

(a) $f'(2.25) \approx \dfrac{10-6}{2.5-2} = 8\dfrac{\text{mg/cm}^2}{\text{cm}}$. The density of the bacteria population is increasing at a rate of approximately $8\dfrac{\text{mg/cm}^2}{\text{cm}}$ when moving outward from the center of the petri dish at a distance of 2.25 cm from the center. [1]

(b) $2\pi\int_0^4 rf(r)\,dr \approx 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5) = 269\pi \approx 845.088 \text{ mg}$.

(c) Since $\dfrac{d}{dr}(r\cdot f(r)) = rf'(r) + f(r) > 0$, $rf(r)$ is increasing on [0, 4]. Therefore, the right hand sum is an overestimate.

(d) The average value of $g(r)$ is $\dfrac{1}{4-1}\int_1^4 g(r)\,dr \approx 9.87579$.[2] Solving $g(k) = 9.87579$ gives $k \approx 2.497$.

Notes:

1. The independent variable here is *r*, the distance from the center of the petri dish. So this rate refers to how the density is changing as we move outward from the center of the dish (that is, as *r* increases).
2. Do not round to three decimal places at this point.

Question AB-2

(a) Let $x_P(t)$ and $x_Q(t)$ be the positions of particles P and Q, respectively, at time t.[1]
Then $x_P(1) = 5 + \int_0^1 v_P(t)\,dt \approx 5.371$ and $x_Q(1) = 10 + \int_0^1 v_Q(t)\,dt \approx 8.564$.

(b) $v_P(1) \approx 0.841 > 0$ and $v_Q(1) = -1 < 0$. At $t = 1$, particle Q is to the right of particle P and moving to the left. At the same time, particle P is moving to the right. Therefore, the particles are moving toward each other.

(c) At $t = 1$, the acceleration of particle Q is $v_Q'(1) \approx 1.027 > 0$. Since $v_Q(1) < 0$ and $v_Q'(1) > 0$, the speed of particle Q is decreasing.

(d) The total distance traveled is $\int_0^{\pi} \left|v_P(t)\right| dt \approx 1.931$.

Notes:

1. It will be helpful to enter and save these two functions in the calculator.

Question AB-3

(a) The x-intercepts of the graph are at $x = 0$ and $x = 2$. Area= $\int_0^2 6x\sqrt{4-x^2}\,dx$. Let

$$u = 4 - x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow -3du = 6xdx.$$

$$\int 6x\sqrt{4-x^2}\,dx = -3\int u^{1/2}du = -3\cdot\frac{u^{3/2}}{3/2} = -2\left(4-x^2\right)^{3/2}.$$ So

$$\int_0^2 6x\sqrt{4-x^2}\,dx = -2\left(4-x^2\right)^{3/2}\Big|_0^2 = 0-(-2)(4)^{3/2} = 16.$$ [1]

.

(b) The radius is a maximum where $\frac{dy}{dx} = \frac{c\left(4-2x^2\right)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$. So

$$1.2 = c\sqrt{2}\sqrt{4-2} = 2c \;\Rightarrow\; c = \frac{1.2}{2} = 0.6.$$ [2]

(c) The volume of this toy is

$$\pi\int_0^2\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2\int_0^2 x^2\left(4-x^2\right)dx = \pi c^2\int_0^2\left(4x^2-x^4\right)dx = .$$

$$\pi c^2\left(\frac{4x^3}{3}-\frac{x^5}{5}\right)\Bigg|_0^2 = \pi c^2\left(\frac{32}{3}-\frac{32}{5}\right).$$

$$2\pi = \pi c^2\left(\frac{32}{3}-\frac{32}{5}\right) \;\Rightarrow\; c^2 = \frac{15}{32} \;\Rightarrow\; c = \sqrt{\frac{15}{32}}.$$ [3]

Notes:

1. Or:

$$u = 4 - x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow -3du = 6xdx.$$

$$\int_0^2 6x\sqrt{4-x^2}\,dx = -3\int_4^0 u^{1/2}du = -3\cdot\frac{u^{3/2}}{3/2}\Bigg|_4^0 = 16.$$

2. You can leave it at $c = \frac{1.2}{2}$.

3. Or just write $c = \sqrt{\dfrac{2}{\left(\frac{32}{3}-\frac{32}{5}\right)}}$.

Question AB-4

(a) $G'(x) = f(x)$, so the graph of G is concave up on $(-4,-2)$ and $(2,6)$ because $G'(x)$ is increasing there.

(b) $G(3) = \int_0^3 f(x)dx = \frac{3}{2} - \frac{3}{2} - 3\frac{1}{2} = -3.5$; $f'(3) = 1$.
$P'(3) = G(3)\cdot f'(3) + G'(3)\cdot f(3) = -3.5\cdot 1 + (-3)\cdot(-3) = 5.5$.

(c) Since f is continuous, the Fundamental Theorem tells us G is differentiable, and therefore G is continuous. So $\lim_{x\to 2} G(x) = G(2) = 0$. Also, $\lim_{x\to 2}(x^2 - 2x) = 0$.
L'Hospital's Rule applies: $\lim_{x\to 2}\frac{G(x)}{x^2-2x} = \lim_{x\to 2}\frac{G'(x)}{2x-2} = \frac{-4}{2} = -2$.

(d) The average rate of change is $\frac{G(2)-G(-4)}{2-(-4)} = \frac{0-(-16)}{6} = \frac{8}{3}$. From Part (c) we know that G is differentiable and continuous, so the conditions of the Mean Value Theorem are satisfied. [1] Therefore, there is a value of c such that $-4 < c < 2$ and $G'(c) = \frac{8}{3}$.

Notes:

1. It is important to state these necessary conditions for the MVT.

Question AB-5

(a) $4y \cdot \frac{dy}{dx} = y\cos x + \sin x \cdot \frac{dy}{dx} \Rightarrow (4y - \sin x) \cdot \frac{dy}{dx} = y\cos x \Rightarrow \frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$.

(b) At $(0, \sqrt{3})$, $\frac{dy}{dx} = \frac{\sqrt{3} \cdot 1}{4\sqrt{3} - 0} = \frac{1}{4}$. The tangent line is given by $y - \sqrt{3} = \frac{x}{4}$. [1]

(c) $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{2}$; then $2y^2 - 6 = y \cdot 1 \Rightarrow 2y^2 - y + 6 = 0 \Rightarrow (2y+3)(y-2) = 0$. $y = 2$ satisfies $0 \le y \le \pi$. The point with a horizontal tangent line with $0 \le y \le \pi$ is $\left(\frac{\pi}{2}, 2\right)$.

(d) Since $y\left(\frac{\pi}{2}\right) = 2$ and *y* is continuous, there is an open interval containing $x = \frac{\pi}{2}$ where *y* remains close to 2. In that interval $4y - \sin x > 0$ so the sign of $\frac{dy}{dx}$ is determined by $\cos x$. Just to the left of $\frac{\pi}{2}$, $\frac{dy}{dx} > 0$ and just to the right of $\frac{\pi}{2}$, $\frac{dy}{dx} < 0$. Therefore, *f* has a local maximum at this point. [2]

Notes:

1. Or $y = \frac{x}{4} + \sqrt{3}$.

2. We could also have used the second derivative test and show that at $\left(\frac{\pi}{2}, 2\right)$, $\frac{d^2y}{dx^2} < 0$:

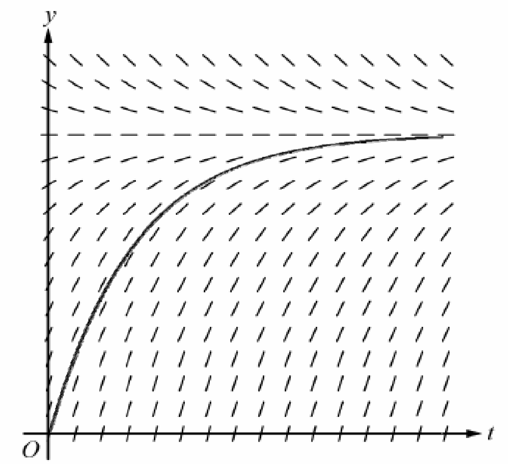
$$\frac{d^2y}{dx^2} = \frac{(4y - \sin x)\left(-y\sin x + \cos x \cdot \frac{dy}{dx}\right) - y\cos x\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$$

. At $\left(\frac{\pi}{2}, 2\right)$,

$$\frac{d^2y}{dx^2} = \frac{(8-1)(-2 \cdot 1 + 0) - 0}{(8-1)^2} = -\frac{14}{49} < 0.$$

Question AB-6

(a)



(b) $\lim_{t\to\infty} A(t) = 12$ means that as time t gets infinitely large, the amount of medication in the patient approaches 12 milligrams.

(c) $\frac{dy}{12-y} = \frac{1}{3}dt \Rightarrow -\ln|12-y| = \frac{t}{3} + C$. From the initial condition $A(0) = 0$,

$C = -\ln 12$. We get $-\ln|12-y| = \frac{t}{3} - \ln 12 \Rightarrow \ln|12-y| = \ln 12 - \frac{t}{3} \Rightarrow$

$|12-y| = e^{\ln 12 - t/3} = 12e^{-t/3}$. Since $y = 0$ at the initial condition,

$|12-y| = 12 - y \Rightarrow A(t) = 12 - 12e^{-t/3}$.

(d) $\frac{d^2y}{dt^2} = -\frac{(t+2)\frac{dy}{dt} - y}{(t+2)^2} = -\frac{(t+2)\left(3 - \frac{y}{t+2}\right) - y}{(t+2)^2}$. At the point $(1, 2.5)$,

$\frac{d^2y}{dt^2} = -\frac{(3)\left(3 - \frac{2.5}{3}\right) - 2.5}{(3)^2} < 0$, so the rate of change is decreasing. [1]

Notes:

1. A common mistake would be to answer whether y is increasing or decreasing, not the rate of change of y.

2021 BC
AP Calculus Free-Response
Solutions and Notes

Question BC-1

See AB Question 1.

Question BC-2

(a) The speed is $\sqrt{\left((1.2-1)e^{1.2^2}\right)^2+\left(\sin\left(1.2^{1.25}\right)\right)^2}$ and the acceleration vector is $\langle x''(1.2),\, y''(1.2)\rangle \approx \langle 6.247, 0.405\rangle$.

(b) The total distance traveled is $\int_0^{1.2}\sqrt{\left((t-1)e^{t^2}\right)^2+\left(\sin\left(t^{1.25}\right)\right)^2}\,dt \approx 1.010$.

(c) $x'(t)=0$ at $t=1$. For $0\le t<1$, $x'(t)<0$ and for $1<t<\infty$, $x'(t)>0$. Therefore, $x(t)$ has a minimum at $t = 1$; the particle is farthest to the left at $t = 1$.
$x(1)=-2+\int_0^1 (t-1)e^{t^2}\,dt \approx -2.604$ and $y(1)=5+\int_0^1 \sin\left(t^{1.25}\right)dt \approx 5.410$.
There is no point where the particle is farthest to the right because the x-coordinate of the particle increases for $t>1$. [1]

Notes:

1. It would be possible for the particle's x-coordinate to increase for all $t > 1$ and still have a rightmost position of −2. However, finding that the position is greater than −2 at some time after $t = 1$ excludes that possibility.

Question BC-3

See AB Question 3.

Question BC-4

See AB Question 4.

Question BC-5

(a) $f'(1)=4\cdot 1\ln 1=0$, so the second degree Taylor Polynomial at $x=1$ is

$P_2(x)=4+\frac{4}{2!}(x-1)^2=4+2(x-1)^2$. $f(2)\approx P_2(2)=4+2(2-1)^2=6$.

(b) At $(1,4)$, $\frac{dy}{dx}=0$ and so $f(1.5)\approx 4+0\cdot 0.5=4$. At $(1.5,4)$, $\frac{dy}{dx}=4\cdot\frac{3}{2}\ln\left(\frac{3}{2}\right)=6\ln\left(\frac{3}{2}\right)$,

so $f(2)\approx 4+6\ln\left(\frac{3}{2}\right)\cdot 0.5=4+3\ln\left(\frac{3}{2}\right)$.

(c) $\frac{1}{y}dy=x\ln x dx \Rightarrow \ln|y|=\int x\ln x\,dx$. Using integration by parts with

$u=\ln x, dv=xdx \Rightarrow du=\frac{1}{x}dx, v=\frac{x^2}{2}$, we get

$\int x\ln x\,dx=\frac{x^2\ln x}{2}-\int\frac{x^2}{2}\cdot\frac{1}{x}dx=\frac{x^2\ln x}{2}-\frac{1}{2}\int x\,dx=\frac{x^2\ln x}{2}-\frac{x^2}{4}$. So

$\ln|y|=\frac{x^2\ln x}{2}-\frac{1}{4}x^2+C$. Using the initial condition,

$\ln 4=\frac{1^2\ln 1}{2}-\frac{1}{4}1^2+C\Rightarrow C=\ln 4+\frac{1}{4}$. $\ln|y|=\frac{x^2\ln x}{2}-\frac{x^2}{4}+\ln 4+\frac{1}{4} \Rightarrow |y|=4e^{\frac{x^2\ln x}{2}-\frac{x^2}{4}+\frac{1}{4}}$.

Since $y>0$ at the initial condition, the solution is $f(x)=4e^{\frac{x^2\ln x}{2}-\frac{x^2}{4}+\frac{1}{4}}$.

Question BC-6

(a) The function e^{-x} is continuous, positive, and decreasing for all $x \geq 0$.
$\int_0^{\infty} e^{-x} dx = \lim_{a \to \infty} \int_1^a e^{-x} dx = \lim_{a \to \infty} \left(-e^{-x}\right)\Big|_0^a = \lim_{a \to \infty}\left(-e^{-a} - (-1)\right) = 1$. Since the integral converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) $\lim_{n \to \infty} \frac{\frac{1}{2e^n + 3}}{\frac{1}{e^n}} = \lim_{n \to \infty} \frac{1}{2 + \frac{3}{e^n}} = \frac{1}{2}$. Since $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{2e^n + 3}$ also converges, and the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

(c) $\lim_{n \to \infty} \left| \frac{\frac{x^{n+1}}{2e^{n+1} + 3}}{\frac{x^n}{2e^n + 3}} \right| = \lim_{n \to \infty} \left| \frac{x \cdot \left(2e^n + 3\right)}{2e^{n+1} + 3} \right| = |x| \cdot \lim_{n \to \infty} \frac{2e^n}{2e^{n+1}} = \frac{|x|}{e} < 1 \Rightarrow -e < x < e$, so the radius of convergence is *e*.

(d) The error bound is found using the third term, which is when $n = 2$. The error bound is $\frac{1}{2e^2 + 3}$. [1]

Notes:

1. The Alternating Series Error Bound can be used since the terms of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ alternate in sign, and their magnitude decreases to 0. Therefore the series converges by the Alternating Series Test.