Mathematics
for the Digital Age

and

Programming
in Python

>>> Second Edition:
with Python 3

Maria Litvin
Phillips Academy, Andover, Massachusetts

Gary Litvin
Skylight Software, Inc.

Skylight Publishing
Andover, Massachusetts

Skylight Publishing
9 Bartlet Street, Suite 70
Andover, MA 01810

web: http://www.skylit.com
e-mail: sales@skylit.com
support@skylit.com

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8 Counting

8.1 Prologue

Combinatorics is a branch of mathematics that deals with counting all possible combinations, arrangements, sequences, or sets of objects. For example: Ice cream comes in two sizes, three flavors, and with any one of five toppings; in how many different ways can you order an ice cream? It is often not feasible simply to list and count all possible arrangements or combinations; combinatorics offers more sophisticated methods of counting. These methods are not very complicated: they basically use the four arithmetic operations. The trick is to know when to multiply, when to divide, when to add, and when to subtract.

Counting methods came to prominence in the 17th century when Blaise Pascal (1623-1662) and others got interested in computing the odds in gambling games. For example, what are the odds of getting 11 when we roll two dice? To find out, we need to know the number of all favorable outcomes (possible rolls that give 11) and the number of all possible outcomes. Using combinatorial counting techniques, we can analyze the likelihood of different arrangements and outcomes in card games such as Poker and Blackjack, dice games such as Craps, and so on. We can also analyze the complexity of computer algorithms and the running time and required space for computer programs.

8.2 The Multiplication Rule

The key operation in combinatorics is multiplication.

> **Suppose an object is described by two independent attributes. The total number of possible combinations of their values is equal to the number of possible values for the first attribute times the number of possible values for the second attribute.**

The multiplication rule is illustrated in Figure 8-1.

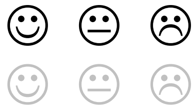

Figure 8-1. 3 shapes of the mouth and 2 colors make 6 possible faces.

If an object has three, four, or k attributes, then we multiply three, four, or k numbers, respectively.

Example 1

How many two-digit positive integers are there?

Solution

We can choose the tens digit in 9 ways (1 through 9) and the units digit in 10 ways. The answer is $9 \cdot 10 = 90$.

Example 2

A license plate has three digits followed by three letters. All combinations of digits and letters are allowed. What is the total number of possible license plates?

Solution

There are 10 ways to choose the first digit, 10 for the second digit, and 10 for the third digit. There are 26 ways to choose the first letter; same for the second letter and the third letter. The answer is $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 17{,}576{,}000$.

Example 3

How many different values can be represented in one byte?

Solution

There are 2 ways to set the first bit, 2 ways to set the second bit, and so on. The answer is $2^8 = 256$.

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The multiplication rule explains how very large numbers can easily arise in combinatorial problems — or why it is so hard to win the lottery.

Example 4

How many ways are there to cover a whole 19 by 19 Go board with white and black stones?

Solution

2^{361} — more than 2×10^{120}, that is many times more than there are atoms in the universe.

Exercises

1. A combination lock has three wheels with the digits 0 through 9 on each wheel. What is the total number of possible combinations? ✓

2. Ice cream comes in two sizes, three flavors, and with any one of five toppings. In how many different ways can you order an ice cream?

3. How many three-letter names are possible in Python, such that all letters are in lower case, the middle letter is a vowel (‘a’, ‘e’, ‘i’, ‘o’, ‘u’) and the other two are consonants? ✓

4. An experiment consists of tossing a coin 10 times. The outcome is recorded as a string of ten letters, either “H” or “T”. How many different outcome strings are possible?

5.■ In how many ways can we split 10 different marbles between two kids? ⦃ Hint: see Question 4. ⦄

6.■ What is the number of all possible subsets of a set of ten elements, including the empty set and the whole set? ⸨ Hint: see Question 5. ⸩

7. What is the number of all possible colors that can be shown on a computer screen, if the graphics adapter generates a color using 16 bits for each of the red, green, and blue components of the color? ✓

8.■ What is the number of all possible configurations of five disks of different sizes on three pegs if we are not allowed to place a bigger disk on top of a smaller one?

8.3 Permutations

The multiplication rule also applies when we want to count possible arrangements of objects without repetition.

Example 1

In how many ways can a coach choose three players, a point guard, a shooting guard, and a center, from a team of seven players?

Solution

If they came from three different teams, we could choose the first player in 7 ways, the second player in 7 ways, and the third player in 7 ways. But here we cannot just multiply $7 \cdot 7 \cdot 7$ because all three players come from the same team. Once we have chosen the first player, there are only 6 players left. So we can choose the second player in 6 different ways. Once we have done that, there are only 5 players left, so we can choose the third player in 5 different ways. The answer is $7 \cdot 6 \cdot 5$.

Example 2

How many ways are there to seat 6 students in a classroom with 20 desks?

Solution

We can seat the first student at any one of the 20 desks, the second student at any one of the remaining 19 desks, and so on. The answer is $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = 27,907,200$.

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An ordering of n different objects or symbols is called a *permutation*. There are $n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = n!$ possible permutations of n objects.

We can choose the object for the first position in n ways, for the second position in $n-1$ ways, and so on. Only one object remains for the last position.

Recall that the product $1 \cdot 2 \cdot \ldots \cdot n = n \cdot (n-1) \cdot \ldots \cdot 1$ is called "n-factorial." It is denoted by $n!$.

It is possible to arrive at the above result in a different way. Let P_n be the number of permutations of n objects. Let's take n objects, call one of them x and set it aside. There are P_{n-1} permutations of the remaining $n-1$ objects. For each of these P_{n-1} permutations, we can insert x at the beginning, between any two objects, or at the end, thus creating n different permutations of n objects. So, from each of the P_{n-1} permutations of n-1 objects, we now obtained n different permutations of n objects. Therefore, $P_n = nP_{n-1}$. Also, $P_1 = 1$, because there is only one permutation of 1 object. It is now clear that $P_n = n!$. To complete the proof more formally we would need to refer to *mathematical induction*, which is explained in Chapter 11.

Example 3

How many ways are there to put 5 different party hats on 5 people?

Solution

5! = 120.

Example 4

How many ways are there to arrange the cards in the deck of 52 cards?

Solution

52! = 80658175170943878571660636856403766975289505440883277824000000000000.

Exercises

1. How many three-digit integers have all different digits? ✓

2.■ How many five-letter "words" (strings of letters) do not have the same letter twice in a row? Count any combination of letters as a "word."

3. How many ways are there to choose "most likely to succeed," "best athlete," "best dancer," "best dressed," and "class clown," a boy and a girl in each category, in a graduating middle school class of 30 girls and 36 boys? The same person cannot be chosen in two categories. ✓

4. How many ways are there to seat 6 people in a row of 8 chairs? ✓

5.■ How many ways are there to seat 8 people in a row of 6 chairs, with two people left out? { Hint: assign chairs to people, not people to chairs. } ✓

6. How many pages will be required to print all possible permutations of letters "ABCDEFGHIJKL"? Each permutation is printed on a separate line; 60 lines fit on a page.

7.■ Amelia is solving the cryptarithmetic puzzle

```
  S E N D
+ M O R E
---------
M O N E Y
```

Her method is to try all possible substitutions of different digits for different letters (excluding 0 for 'M' and 'S', because a number cannot start with a 0). If she takes one minute for each try (her addition skills are excellent) and finds the answer after trying half of all possible substitutions, how long will it take Amelia to solve the puzzle? ✓

8.■ Two words are called *anagrams* of each other if they are made up of the same letters used in a different order: for example, MANGO and AMONG. Jesse decided to write a program that finds all anagrams of a given word by generating all possible permutations of its letters and looking up each permutation in a list of words (obtained in a file). If the program can generate and look up 360 permutations per second, how long will it take it to find all anagrams of "BINARY"? "DECIMAL"? What about "CONVERSATION"? ✓

8.4 Using Division

In many combinatorial problems it is easier first to overcount, counting each arrangement several times. This is fine, as long as we count each arrangement the same number of times and know that number. Then we can get the answer by dividing our total count by the number of times we counted each arrangement.

Example 1

n teams are playing in a round-robin tournament; each team plays once with every other team. How many games will be played?

Solution

There are n ways to choose the first team and $(n-1)$ ways to choose the second team for a game. That gives $n(n-1)$. However, this way we have counted each game twice, because it doesn't matter which team is called "first" and which "second." If we swap the two teams we get the same game. Therefore, the answer is $\frac{n(n-1)}{2}$.

Example 2

How many words can we make from the letters A L A B A M A (assuming any arrangement of letters is a "word")?

Solution

If all seven letters were different, we would have 7·6·5·4·3·2·1 = 7! words. But the four A's are the same, so we get the same word when we permute them. We have counted each arrangement of seven letters the number of times equal to the number of permutations of four A's, that is 4! times. The answer is $\frac{7!}{4!} = 5 \cdot 6 \cdot 7 = 210$.

Example 3

There are five identical pairs of gloves in a drawer. If you pull out two gloves at random, what are the odds that they will make a pair?

Solution

We can pull the first glove in 10 ways and the second glove in 9 ways. However, if the order is not important, we have counted each possibility twice. So the total number of ways to choose two gloves is $\frac{10 \cdot 9}{2} = 45$. We can pull a left glove in 5 ways and a right glove in 5 ways, so the number of ways to pull out a pair is $5 \cdot 5 = 25$. The number of ways to pull out 2 left gloves is $\frac{5 \cdot 4}{2} = 10$. The same for two right gloves. The number of ways to pull out two gloves that don't make a pair is $10 + 10 = 20$. So the numbers match: $25 + 20 = 45$. The odds of getting a pair are $25 : 20 = 5 : 4$ (we say, "5 to 4").

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Sometimes, a problem can be solved either by division or by "anchoring" specific properties of an object or an arrangement.

Example 4

How many ways are there to seat a teacher and six students at a round table? Arrangements are considered the same as long as each person has the same left and right neighbors.

Solution

We can say that there are 7! total arrangements and 7 ways to rotate the table, so the answer is equal to $\frac{7!}{7} = 6! = 720$.

Another solution may be obtained by sitting down the teacher anywhere at the table; then there are 6! ways to arrange the students.

Exercises

1. In how many ways can we choose two puppies out of a litter of 12 puppies? ✓

2. Ice cream comes in two sizes, three flavors, and with any <u>two</u> of five toppings. How many different orders are possible? ✓

3.■ How many words can we make from the letters T E N N E S S E E (assuming any arrangement of letters is a "word")? ✓

4.■ How many ways are there to seat a group of six at a rectangular dinner table? The host and the hostess must sit at the short sides and the four guests two at each of the longer sides. Arrangements are considered the same as long as each guest has the same neighbors.

5.■ In how many ways can we color the six faces of a cube in six different colors? The cube can be rotated any way you want — the colorings are considered the same. ✓

6.◆ An airline is planning three non-stop flights from the East Coast of the United States to the Caribbean. The airline serves six major cities on the East Coast and 12 different Caribbean islands. Two non-stop flights can go to the same island, but they cannot originate at the same East Coast city. How many different configurations of the three flights are possible? ✓

7.◆ How many ways are there to place eight rooks on a chessboard so that none of them threatens any other? A rook moves vertically or horizontally by any number of squares. (The 64 squares on a chessboard are marked "a1" through "h8" and considered different; the eight rooks are considered identical.) ✓

8.5 Combinations

Combinations are selections of k elements from a given set of n elements ($0 \le k \le n$), disregarding the order. This number is written $\binom{n}{k}$ and read "n-choose-k."

$\binom{n}{k}$ are important numbers in mathematics and they have many nice properties. First of all, notice that there is a kind of symmetry: $\binom{n}{k} = \binom{n}{n-k}$. Indeed, to choose k objects is the same as to set aside the remaining $n-k$ objects. Note that $\binom{n}{n} = 1$. Mathematicians have agreed that $\binom{n}{0} = 1$, so the symmetry is complete.

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Let's derive a formula (with factorials) for n-choose-k using the multiplication and division rules. Suppose we choose k objects as follows: we first arrange all n objects in line, then take the first k. The total number of arrangements is $n!$. However, if we rearrange the first k objects and/or the remaining $n-k$ objects, we will end up with the same selection. There are $k!$ ways to rearrange the first k objects and $(n-k)!$ ways to rearrange the remaining $n-k$ objects. So we have counted each selection $k!(n-k)!$ times. Therefore,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1}$$

It is convenient to define 0! as 1, so that the formula works for $k = 0$, too.

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It is useful to remember the formulas for $\binom{n}{k}$ for $k = 0$, 1, 2, and 3:

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Example 1

How many ways are there to split 4 different pencils between two children, so that each gets two pencils?

Solution

We need to choose two pencils for the first child — the second child gets the other two.

$$\binom{4}{2} = \frac{4 \cdot 3}{2} = 6 .$$

Example 2

How many ways are there to choose 5 cards from a deck of 52 cards?

Solution

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960 .$$

Example 3

How many different bit patterns in a byte have exactly three bits set?

Solution

$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$.

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$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is a neat formula, and it confirms the symmetry $\binom{n}{k} = \binom{n}{n-k}$. It is not practical for some computations, though, because factorials get very large quickly. For computer programs, it is more convenient to rewrite $\binom{n}{k}$ as a product of fractions:

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \ldots \cdot \frac{n-k+1}{1}$$

Exercises

1. How many ways are there to order a pizza with three toppings out of five possible toppings? ✓

2. Write $\binom{4}{k}$ for k = 0, 1, 2, 3, 4.

3. Pat has 40 beads of which 35 are white and 5 are black. The beads are identical, except for the color. How many different bracelets can Pat make using all 40 beads? The ends of a bracelet are asymmetrical: one has a hook and the other an eyelet. ✓

4.■ Write a Python function `nChooseK` that calculates and returns $\binom{n}{k}$ as an `int`. The function should work for $0 \le k \le n$. ⊰ Hint: use a `float` for the product of fractions, then convert it into an `int`, using the built-in function `round`. Rounding helps avoid tiny inaccuracies in computer arithmetic. ⊱

5.■ How many ways are there to split 9 different stickers among three people, three apiece? ✓

6.■ How many different tic-tac-toe grids have 3 X's and 3 O's? ⊰ Hint: see Question 5. ⊱

7.◆ How many ways are there to make a "full house" Poker hand from a deck of 52 cards? "Full house" is three cards of one rank and a pair of another rank. ✓

8.◆ If you multiply out $\underbrace{(x+1)\cdot\ldots\cdot(x+1)}_{n\text{ times}}$, what is the coefficient at x^k in the resulting polynomial?

9.◆ Consider the following Boolean expression:

$$((P \wedge Q) \vee \neg(P \vee Q)) \wedge ((P \wedge \neg Q) \vee \neg P)$$

It has three $\wedge$ and three $\vee$ operators. Suppose we reshuffle these six operators to make new expressions. Show that among all different expressions obtained this way, at least two will have the same truth tables. ⊰ Hint: don't do logic — just count. ⊱

8.6 Using Addition and Subtraction

Addition comes into play when it is easier to split the set of all possible objects or arrangements into two or more disjoint sets, count the arrangements in each of these sets separately, and then add the results.

Example 1

How many ways are there to choose one or two ice cream toppings from five available toppings?

Solution

One topping is not the same as two toppings! There are 5 ways to choose one and $\frac{5 \cdot 4}{2} = 10$ ways to choose two. The answer is 5 + 10 = 15.

Example 2

How many ways are there to schedule two quizzes in a five-day week, if the quizzes cannot be on the same day or on consecutive days?

Solution

If the first quiz is on the first day, the second quiz can be on the third, fourth or fifth day — three possibilities. If the first quiz is on the second day, the second quiz can be on the fourth or fifth day — 2 possibilities. If the first quiz is on the third day, the second quiz must be on the fifth day — 1 possibility. The answer is 3 + 2 + 1 = 6.

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If you have trouble counting objects or arrangements that satisfy a certain condition, you may try to count <u>all</u> possible arrangements, then count the arrangements that <u>do not</u> satisfy the condition and subtract their number from the total.

Example 3

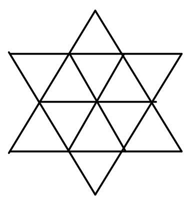
How many ways are there to seat two adults and four kids in a row in a movie theater so that the two adults are not next to each other?

Solution

The total possible number of arrangements is 6! = 720. Let's count the number of arrangements where the adults are sitting together. The number of ways to choose two seats together for the adults is 5; the number of ways to seat the adults in these seats is 2; the number of ways to seat the kids in the remaining four seats is 4!. So the number of arrangements where the adults sit together is $5 \cdot 2 \cdot (4!) = 240$. The answer is $720 - 240 = 480$.

Exercises

1. How many triangles are there in the following picture?



⋞ Hint: count triangles of different sizes separately. ⋟

2. How many different ways are there to make 50 cents using quarters, dimes, and nickels? ✓

3. How many pairs of integers *m* and *n*, such that $2 \le m, n \le 12$, have no common factors (except 1)? ✓

4. Recall that a valid name in Python can consist of upper- and lowercase letters, digits, and underscore characters, but cannot start with a digit. How many valid Python names of length 3 or less are there? ✓

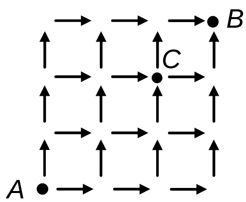
5.■ The Andover co-ed indoor soccer league tournament requires a 7-member team with at least two women. The Andover Pythons club has 8 men and 5 women. How many ways do the Pythons have to form a tournament team?

6.◆ Given cards 2 through 10 in four suits (36 cards total), how many ways are there to make 21 on three cards? ⋞ Hint: consider separately the cases where all three cards have the same rank, two cards have the same rank, and all three cards have different ranks. ⋟ ✓

7. How many four-digit numbers have at least one 7 among their digits?
⋞ Hint: count the numbers that do not have any 7s. ⋟ ✓

8. How many positive integers below 1000 are not divisible by 6?

9.◆ In the diagram below, how many different paths lead from *A* to *B*?



How many of them do not go through point *C*? ✓

10.◆ A password can contain upper- and lowercase letters as well as digits; it must include at least one uppercase letter and at least one digit. How many different four- or five-character passwords are possible? ✓

8.7 Review

Terms and notation introduced in this chapter:

Combinatorics
Multiplication rule
Permutations
Factorial
Combinations
n-Choose-k

$$1 \cdot 2 \cdot \ldots \cdot n = n!$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Some of the Python features introduced in this chapter:

```
round(x)
```