Mathematics
for the Digital Age

and

Programming
in Python

>>> Second Edition:
with Python 3

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16 More Graphs

16.1 Prologue

In this chapter we continue talking about graphs. Graph theory is a vast and fascinating branch of mathematics; even a very brief and superficial survey did not fit into one chapter. The basic definitions are simple, making it a great playground for amateur mathematicians of all ages. But some of the theorems are very hard.

We will first discuss representation of graphs using adjacency matrices, then talk about coloring geographic maps and planar graphs, which, as we will see, is the same thing, and finally try to tackle the Four Color Theorem, for planar graphs.

But first let us recall some of the terms from the previous chapter:

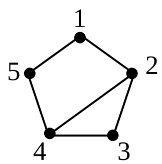
- In a *simple graph*, at most one edge connects any two vertices, and a vertex cannot be connected to itself.
- In a *multigraph*, more than one edge can connect two vertices, and a vertex can be connected to itself.
- In a *directed graph*, edges have directions; they are represented by arrows.
- In a *weighted graph*, a weight (a real number) is assigned to each edge.
- In a *connected graph*, any two vertices are connected by a path.
- C_n is a simple graph with n vertices that consists of one cycle. K_n is a simple graph with n vertices in which every pair of vertices is connected by an edge.
- The degree of a vertex is the number of edges that come out of that vertex.

16.2 Adjacency Matrices

Suppose we have a simple graph with n vertices.

Two vertices connected by an edge are called *adjacent*.

One way to describe the graph's edges is simply to list all pairs of adjacent vertices. But there is another way. We can make a table with n rows and n columns and put a checkmark in the intersection of the *i*-th row and *j*-th column if the *i*-th vertex is connected to the *j*-th vertex. In math and computer programs it is more convenient to use 0s and 1s instead of checkmarks: 1 indicates an edge and 0 no edge. A square matrix that describes the edges of a graph is called the *adjacency matrix* of that graph (Figure 16-1).



$\Longrightarrow$

	1	2	3	4	5
1		✓			✓
2	✓		✓	✓	
3		✓		✓	
4		✓	✓		✓
5	✓			✓	

$\Longrightarrow$

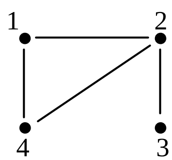
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 16-1. Representing a graph by its adjacency matrix

For a simple (not directed) graph, its adjacency matrix contains only 0s and 1s, is symmetrical over the main diagonal, and has zeros on the diagonal.

Example 1

Write the adjacency matrix for the following graph:



Solution

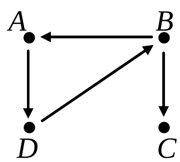
$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

❖ ❖ ❖

For a directed graph, we can agree that the value a_{ij} in its adjacency matrix is 1 if there is an arrow from the *i*-th vertex to the *j*-th vertex. The adjacency matrix for a directed graph is not necessarily symmetrical.

Example 2

Write the adjacency matrix for the following directed graph:



Solution

Assuming the vertices are numbered in order *A*, *B*, *C*, *D*:

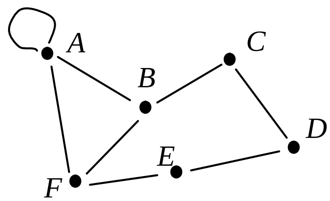
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

❖ ❖ ❖

For a multigraph, an element of the adjacency matrix holds the number of edges between the corresponding vertices. For a simple weighted graph, instead of 1s and 0s we can put into the matrix the weights assigned to the edges.

Exercises

1. Write the adjacency matrix for the following graph: ✓



2. Draw a directed graph that corresponds to the following adjacency matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

3. Write adjacency matrices for C_5 and K_5. ✓

4.■ Which of the following operations on the adjacency matrix for a directed graph always results in a matrix for an isomorphic graph? (Isomorphism for graphs is defined in Section 15.3.) ✓

(a) Flipping the matrix symmetrically over the main diagonal
(b) Swapping any two rows
(c) Swapping any two columns
(d) Swapping the *i*-th and *j*-th rows, then the *i*-th and *j*-th columns (for any *i* and *j*).

5.■ Write and test a Python function that takes an adjacency matrix for a simple graph and returns a list of its edges. An edge that connects the *i*-th and *j*-th vertices should be described by the tuple `(i,j)`, where $i < j$.

6.■ Modify the function from Question 5 so that it works for directed graphs. An arrow from the *i*-th to the *j*-th vertex should be described by the tuple `(i,j)`.

7.◆ Write and test a Python function that takes a simple graph $G = (V, E)$ (described by two sets, V and E) and returns its adjacency matrix.

8.◆ Suppose A is an adjacency matrix for a directed graph. How can you interpret the values of the elements of $A^2 = A \cdot A$ in terms of existing paths in the graph? ✓

9.◆ Write and test a Python function `allPaths(g,k)` that takes a directed graph `g` (described by two sets, V and E) and a positive integer k and calculates the number p_{ij} of paths of length k from the i-th vertex to the j-th vertex for all i and j. The result should be returned as a matrix with values p_{ij}. Hint: see Questions 7 and 8.

16.3 Coloring Maps

In a geographical map, neighboring regions, countries, or states are often shown in different colors. The map in Figure 16-2 uses five "colors" (five shades of gray, plus white for water).

Figure 16-2. A map colored in five "colors"

Can any map be colored in five colors? Could we use fewer colors? Four colors? Three colors? These questions have little to do with making maps: we could use as many colors as we need to make a pretty map. Instead, these questions end up in the realm of mathematics, as many interesting questions do.

Once we enter the realm of mathematics, though, we must be very precise. What is a map? What is a "country"? What does "neighboring countries" mean? Do "countries" that only touch in one point, like Colorado and Arizona at The Four Corners on a U.S. map, share a border? Can a country be split into two disjoint regions, like Alaska and the mainland United States? Can a country be an island? Several islands? Can a country be entirely inside another country, like San Marino in Italy?

We need formal definitions, and in this case they get rather messy. Let's assume that each "country" is one contiguous region. Pick a "capital city" in each country. For each pair of neighboring countries, build a "road" to connect their capital cities in such a way that the road crosses a segment of the shared border and stays entirely within the area of the two countries. Do not allow two roads to cross each other. What you get is a *planar graph* — a graph drawn on the plane (Figure 16-3). The edges do not have to be straight line segments, but they cannot intersect.

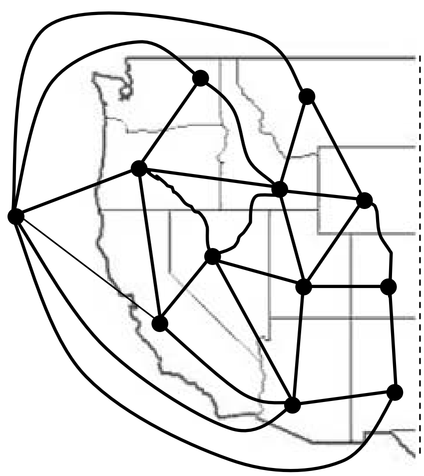


Figure 16-3. A planar graph corresponds to a geographical map

The problem of coloring a map becomes the problem of coloring the vertices of the corresponding planar graph. (It is not practical, of course, to "color" points; we simply assign them colors or numbers or symbols.)

> **A graph is called *properly colored* if any two adjacent vertices are colored in different colors.**

In coloring problems we consider only connected graphs, because if a graph is not connected, we can color each connectivity component separately.

Figure 16-4 shows the graph from Figure 16-3 properly colored in four colors, represented by the numbers 1 through 4.

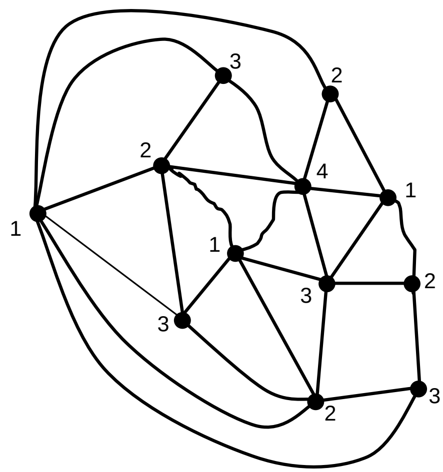
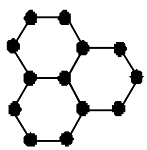


Figure 16-4. A planar graph properly colored in four colors

Exercises

1. Properly color the graph below in two colors. ✓



2. Remove one vertex of degree 2 from the periphery of the graph in Question 1, merging the two edges that come out of it into one. Can the resulting graph be properly colored in two colors?

3. Suppose a graph is properly colored in two colors. What can you tell about the colors of the two vertices that are the endpoints of a path of a certain length? When are they the same? When different?

4. Suppose a graph is properly colored in two colors, and *A* and *B* are two vertices connected by an edge. Consider any two paths from *A* and *B*, respectively, to a third vertex *C*. What can you tell about the parities of the lengths of these paths? Are they even or odd?

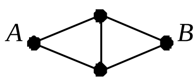
5.■ Formulate the necessary and sufficient condition for a graph to be properly colorable in two colors. State your condition in terms of the absence of certain type of subgraphs in a graph. Justify your answer, by showing that this is indeed a necessary and sufficient condition. ✓

6.◆ Write and test a Python function that colors a given graph in two colors or establishes that it can't be done. Use the following "brute-force" algorithm:

1. Color any one vertex.
2. For each vertex that is already colored, find its neighbors that are not yet colored. Assign to each neighbor the appropriate color.
3. Repeat Step 2 until all the vertices are colored.
4. Check whether the graph is properly colored; return `None` if it isn't.

Assume that the input graph is represented by an *n*-by-*n* adjacency matrix, and return the result as a list of 1s and 2s that holds the colors of the vertices corresponding to the rows of the matrix.

7. Consider the following graph:



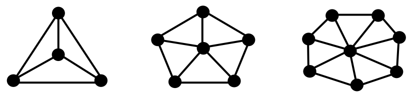
Convince yourself that in any proper coloring of this graph in three colors, the colors of the vertices *A* and *B* are the same. Now, for any $N \geq 2$, give an example of a "longer" graph with similar properties:

a) it is colorable in three colors;

b) it has two vertices *A* and *B* such that the distance between them (the length of the shortest path) is greater than or equal to *N*;

c) in any proper coloring of the graph in three colors, the colors of *A* and *B* are the same.

8.■ None of the following graphs is colorable in three colors:



Clearly, if you take any odd cycle, and connect all its vertices to one "central" vertex, the resulting graph cannot be colored in three colors. Any graph that contains one of these "odd cartwheels" as a subgraph cannot be colored in three colors either. Come up with an example of a graph that does not contain an "odd cartwheel" and still is not colorable in three colors. Hint: see Question 7.

9.◆ Give an example of a graph in three-dimensional space that cannot be colored in three colors and contains no "triangles" (that is cycles of length 3). Hint: you will need at least eleven vertices. ✓

16.4 The Four Color Theorem

The Four Color Conjecture states that any planar graph can be colored in four colors. It first came up in the middle of the 19th century, but the proof had evaded mathematicians for a long time. Many amateurs and professionals tried to show the conjecture to be false by coming up with a counterexample: a map or a graph that cannot be colored in four colors. They failed. Finally, the Four Color Theorem was proved in the late 1970s by Kenneth Appel and Wolfgang Haken. Their proof, published in 1977, was unorthodox: they had to analyze many graph configurations, for which they used a computer program. It required 1200 hours of computer time to complete the proof. (These days it would take less time, of course.) Many mathematicians remained skeptical, though, because they could not verify the proof independently. In 1996 Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas published a shorter and a more manageable proof,* still based on Appel's and Haken's ideas.

In this section we will make a naïve attempt to prove the theorem. We will largely follow the ideas of the British mathematician Alfred Kempe, who proposed his proof at the end of the 1870s. Kempe's proof stood unchallenged for about 10 years, until a major flaw was found in it. We'll go as far as we can with our proof, and see what we can learn from it. At the end you yourself will discover and explain the flaw (see Question 10 in the exercises).

* http://people.math.gatech.edu/~thomas/FC/fourcolor.html

As often happens in proofs of theorems about graphs, we try to use mathematical induction. The idea is to somehow reduce a graph with n vertices to a smaller graph by eliminating one or several vertices. The smaller graph has the desired property by the induction hypothesis. We then try to restore the eliminated vertices in such a way that the property still applies. One has to be very careful doing all this (see Question 5 in the exercises).

One way to reduce the number of vertices in a graph is to "glue together" two vertices. If A and B are two vertices, we can replace them with one vertex O. We connect O to a vertex X with an edge if A or B (or both) are connected to X. (Question 2 in the exercises offers an example of "gluing together" two vertices.) We can glue together three or more vertices in the same way.

Another idea is to split a graph into smaller graphs, then, knowing that each of them has the desired property, combine them back into the original graph, while maintaining the property (see Question 3 in the exercises).

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The edges of a planar graph divide the plane into regions, with one infinite outer region. If a graph is properly colored, and you remove one or several edges, the resulting graph will be properly colored, too. When proving theorems about coloring planar graphs, we can consider only the worst-case scenario, in which no edge can be added to the graph. This happens when all the regions are "triangles" (that is, are bounded by three edges). Such planar graphs are called *fully triangulated*. We can convert any planar graph into a fully triangulated graph by adding a few "diagonals" to every region (see an example in Question 4 in the exercises). If we can color this fully triangulated graph in p colors, then we can color the original graph in p colors, too.

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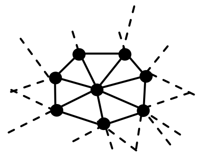
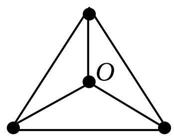
We are now done with the preliminaries and can proceed with our "proof." We will use mathematical induction by the number of vertices in the graph. Clearly any graph with 4 or fewer vertices can be colored in four colors (the base case). Let us take a graph with n vertices, $n > 4$. We assume (induction hypothesis) that any planar graph with fewer than n vertices can be colored in four colors and try to prove that our graph with n vertices can be colored in four colors, too.

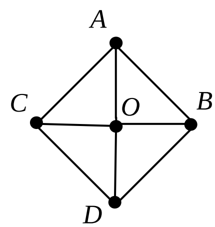
Without loss of generality, we can assume that our graph is fully triangulated. Moreover, if any of the triangles has vertices both inside and outside, the problem of reducing the graph into smaller subgraphs is solved (see Question 3 in the exercises). So the neighborhood of any vertex looks like a simple "cartwheel" with the center at the vertex, and at least three "spokes":



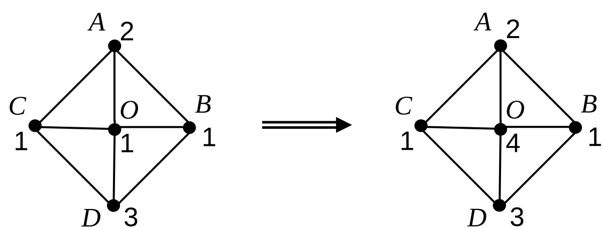
Let's take the vertex *O* in the graph that has the smallest degree. If the degree of *O* is 3, the problem is solved. Indeed, *O* is inside a triangle, and there must be no vertices outside (see Question 3 in the exercises). This means our graph is simply K_4:



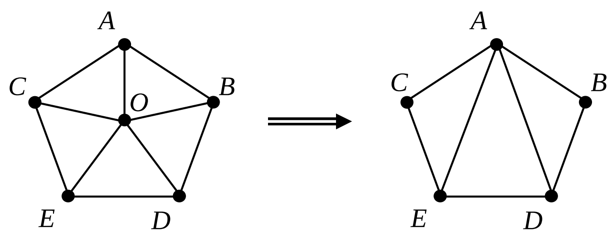
If the degree of *O* is 4, we need a little more work, but not much.



Let us "glue" the vertices *B*, *O*, and *C* together and properly color the resulting graph — we can do that by the induction hypothesis. When we unglue *B*, *O*, and *C*, all three will be colored in the same color, say color 1. The rest of the graph will be properly colored. Vertices *A* and *D* use at most two colors, say 2 and 3. Color 4 remains free, and we can recolor *O* in that color and get a proper coloring of our graph:



It turns out that any planar graph has a vertex of degree 5 or less (see Question 6 in the exercises). We have already considered the cases when the degree of *O* is 3 or 4. The only remaining case is when the degree of *O* is 5. This is the hardest case. If we remove *O* and the edges that connect it to its neighbors and add two "diagonals" to restore full triangulation, we get a smaller graph:



By the induction hypothesis, we can color this smaller graph in four colors. Unfortunately, *A*, *B*, *C*, *D*, and *E*, can use all four colors among them (see Figure 16-5).

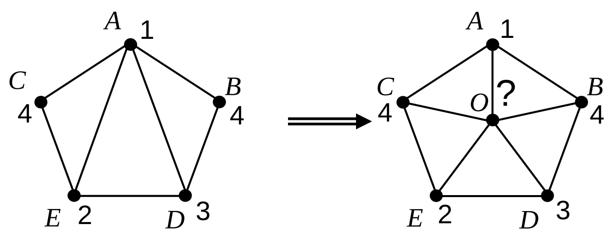


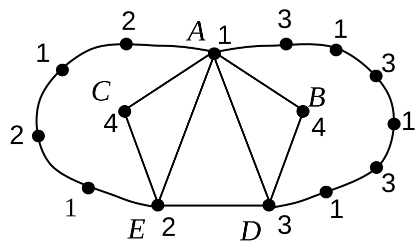
Figure 16-5. The worst case in the Kempe's proof: the neighbors of *O* use all four colors

There is no easy way to simply recolor *O* and get proper coloring. Perhaps we can somehow recolor one of the vertices *A*, *D*, or *E*, or recolor both *B* and *C* to free one of the colors and use it for *O*. But how? That's where Kempe's idea, known as *Kempe's chains*, comes in.

A Kempe's chain is a path in a properly colored graph whose vertices are colored in two colors. Consider a subgraph in a properly-colored graph that consists of all the vertices colored in a pair of colors, say 1 and 2, and all the edges that connect them. This subgraph is not necessarily connected: not all of its vertices are necessarily connected to each other by a 1-2-colored "chain" (path). If this 1-2-colored subgraph is not connected, it splits into several connectivity components. We can take one of these components and flip the colors in it: 1 into 2 and 2 into 1. If our original graph was properly colored, the new coloring will be also proper. This method allows us to recolor some of the vertices in a properly colored graph without disturbing the other vertices.

Let us see how this applies to our proof. If *A* and *E* (see Figure 16-5) are not connected by a 1-2 chain, we can take the largest connected 1-2 subgraph around *A* and flip the colors in it, without disturbing the colors of *B*, *C*, *D*, and *E*. *A* will be recolored from 1 to 2, and 1 will be freed to color *O*. Similarly, if there is no 1-3 chain from *A* to *D*, we can recolor *A* in color 3 and free 1 for *O*.

So far so good. The only situation that remains to consider is when there are both a 1-2 chain from *A* to *E* and a 1-3 chain from *A* to *D*. For example:

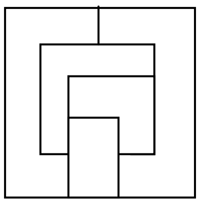


These chains serve as barriers on the plane between other pairs of vertices. The 1-2 chain from *A* to *E* ensures that there is no 4-3 chain from *C* to *D*, because 1-2 and 4-3 chains can't intersect. (If they did intersect in a vertex, what color would that vertex be?) So we can recolor *C* from 4 to 3. Likewise, we can recolor *B* from 4 to 2. 4 will be freed to color *O*.

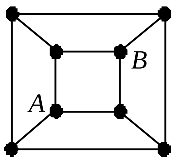
This would complete our proof, if there weren't a big flaw, or, as mathematicians say, a "hole" in it. There is literally a hole in Kempe's chains: he didn't consider what "leaky" barriers they make. The 1-2 and 1-3 chains from *A* can take a round-about path to their destinations instead of the direct path shown above. They can intersect. The 4-2 and 4-3 chains can "leak" out their barriers and get "close" to each other, causing trouble: two vertices on these chains can be connected by an edge. Question 10 in the exercises asks you to provide a counterexample to Kempe's proof.

Exercises

1. The map below has six regions (including the outer region). Draw a corresponding planar graph and properly color in four colors both the map and the graph.



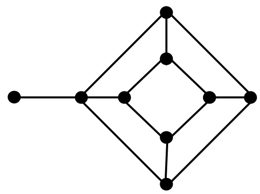
2. Consider a graph



Draw the graph obtained from it by gluing *A* and *B* together. ✓

3.■ Suppose we have a planar graph with n vertices. Suppose we know somehow (for example, from an induction hypothesis) that any planar graph with less than n vertices can be colored in p colors. Suppose also that our graph contains a "triangle" (a region bounded by three edges) with some vertices inside and some vertices outside. Prove that our graph can be colored in p colors, too. ✓

4. Convert the following graph into a fully triangulated graph by adding edges (but not vertices):



Hint: don't forget the infinite outer region: it should be a "triangle," too. ✓

5.◆ One has to be very careful with math induction proofs. Consider the following "proof" of an obviously incorrect statement: Any graph can be properly colored in three colors.

1. <u>Base case</u>. The statement is obviously true for a graph with 3 or fewer vertices.
2. <u>Inductive case</u>. Suppose the statement is true for any graph with less than *n* vertices (inductive hypothesis); let us prove that then the statement is true for any graph with *n* vertices. Let's take a graph with *n* vertices. Let's take any two vertices that are not connected by an edge and glue them together. By the induction hypothesis, we can color the resulting graph in three colors. Now let's unglue the vertices back, preserving the coloring. The vertices that were glued have the same color, but that's OK, since they are not connected by an edge. The original graph is now properly colored, too.

Find a flaw in this "proof." ✓

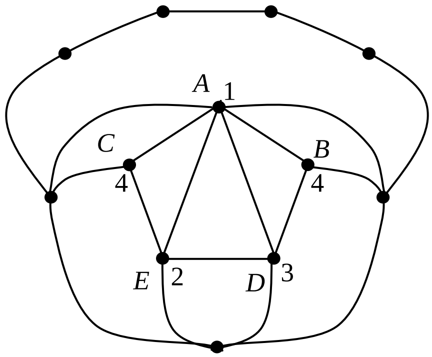
6.◆ Prove that any planar graph has at least one vertex of degree 5 or lower. Hints: it is sufficient to prove this for fully triangulated graphs; use Euler's formula that relates the number of vertices, edges, and regions in a planar graph: $V - E + R = 2$; estimate the number of edges in two ways: from the triangular regions and from the degrees of vertices. ✓

7.◆ Give an example of a fully triangulated planar graph such that all its vertices have a degree of 5 or higher. What is the smallest number of vertices of degree 5 in such a graph? ✓

8. ■ In the graph in Figure 16-4 take the vertex "in Oregon," colored in color 2, find its 2-3 component, and flip the colors in it.

9. ■ Suppose *O* is a vertex in a fully triangulated graph and its degree is greater than or equal to four. Show that there are at least two neighbors of *O* that are not connected by an edge.

10. ◆ Complete the coloring of the graph below to make a counterexample to Kempe's "proof."



11. ◆ Prove the Five Color Theorem: any planar graph can be properly colored in five colors. Hint: use the ideas from the "proof" of the Four Color theorem.

16.5 Review

Terms introduced in this chapter:

Adjacency matrix
Planar graph
Proper coloring
Fully triangulated graph
Four Color Theorem
Kempe's chain
Five Color Theorem